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New Proofs and a Generalisation of Inequalities of Fan, Taussky, and Todd

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Discrete Fourier analysis is used to obtain simple proofs of certain inequalities about finite number sequences determined by Fan, Taussky, and Todd [*Monatsh. Math.* **59** (1955), 73–90] and their converses determined by Milovanović and Milovanović [*J. Math., Anal. Appl.* **88** (1992), 378–387]. Using the same techniques, the inequality

$$\left(2\sin\frac{\pi}{2(n+1)}\right)^4 \sum_{k=1}^{n} b_k^2 \leqslant \sum_{k=0}^{n-1} (b_k - 2b_{k+1} + b_{k+2})^2 \leqslant \left(2 + 2\cos\frac{\pi}{n+1}\right)^2 \sum_{k=1}^{n} b_k^2$$

is proved for all real numbers $0 = b_0, b_1, ..., b_n, b_{n+1} = 0$, which answers a question raised by Alzer [*J. Math. Anal. Appl.* **161** (1991), 142–147]. Second, the method is used to obtain the eigenvalues and eigenvectors of matrices (a_{ij}) that are rotation-invariant, i.e., that obey $(a_{ij}) = (a_{(i+1)(j+1)})$. © 1994 Academic Press, Inc.

1. INTRODUCTION

In their 1955 paper [1], Fan *et al.* prove (among other things) the following inequalities. If $a_1, ..., a_n$ are real numbers, and $a_0 = a_{n+1} = 0$, then

$$\sum_{k=1}^{n+1} (a_k - a_{k-1})^2 \geqslant 2\left(1 - \cos\frac{\pi}{n+1}\right) \sum_{k=1}^{n} a_k^2, \tag{1}$$

with equality if and only if $a_k = c\sin(k\pi/(n+1))$, where c is a real constant. Second, under the hypothesis that $a_1, ..., a_n$ are real and $a_0 = 0$ one has

$$\sum_{k=1}^{n} (a_k - a_{k-1})^2 \geqslant 2\left(1 - \cos\frac{\pi}{2n+1}\right) \sum_{k=1}^{n} a_k^2, \tag{2}$$

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with equality if and only if $a_k = c\sin(k\pi/(2n+1))$, for $k=1, ..., n$. In 1983, Redheffer [4] was able to prove the above inequalities using only elementary calculations.

In 1982 converse inequalities were found by Milovanović and Milovanović [2]: if, again, $a_1, ..., a_n$ are real numbers and $a_0 = a_{n+1} = 0$, then

$$\sum_{k=1}^{n+1} (a_k - a_{k-1})^2 \leqslant 2\left(1+\cos\frac{\pi}{n+1}\right)\sum_{k=1}^{n} a_k^2 \tag{3}$$

with equality if and only if $a_k = c(-1)^{k-1}\sin(k\pi/(n+1))$ for $k=1, ..., n$, and if $a_0 = 0$ and $a_1, ..., a_n$ are arbitrary real numbers then

$$\sum_{k=1}^{n} (a_k - a_{k-1})^2 \leqslant 2\left(1+\cos\frac{2\pi}{2n+1}\right)\sum_{k=1}^{n} a_k^2 \tag{4}$$

with equality if and only if $a_k = c(-1)^{k-1}\sin 2k\pi/(2n+1)$ for $k=1, ..., n$. Their proof is very intricate and difficult to follow. In a recent paper of Alzer [3], a more elementary and shorter proof is given.

In this paper I use yet another method of proving the above inequalities, using discrete Fourier transforms. This method has the asset that it is more intuitive than the previous two, and that it suggests some interesting generalisations.

The referee pointed out to me that the use of finite Fourier analysis for proving inequalities like those above is not new. Schoenberg [5] proved a theorem, using these methods, that is equivalent to Lemma 1, parts a–c (see below), stated in geometrical terms. These and other inequalities were then used to solve several geometrical extremal problems. Another method of proving this result can be found in a paper by Shisha [6]; there the basic tool is a geometric inequality concerning points on an N-sphere.

2. Notation and Definitions

First, some notation. A sequence of complex numbers is written as $(a_k) \in \mathbb{C}^n$, for instance $(e^{2\pi ik/n})$. The index k runs from 0 to $n-1$, where n is the length of the sequence (which will always be clear from the context). To simplify notation, the index k is taken modulo the length, so that $a_0 = a_n$, etc. You can think of the sequence as being infinite but periodic with period n. Sometimes I use the term sequence to mean just an ordered n-tuple of numbers, but if this is the case the numbers will not be written with brackets surrounding them, the latter notation being reserved for periodic sequences only.

The difference operator Δ that operates on a (periodic) sequence of length n is defined by the formula

$$\Delta(a_k) = (a_{k+1} - a_k) = (a_1 - a_0, a_2 - a_1, ..., a_{n-1} - a_{n-2}, a_0 - a_{n-1})$$

and yields another periodic sequence of the same length. Higher powers of the difference operator are defined recursively: $\Delta^2(a_k) = \Delta\Delta(a_k) = (a_k - 2a_{k+1} + a_{k+2})$, etc. By induction, the following expansion of the Δ-operator can be obtained:

$$\Delta^m(a_k) = \left(\sum_{p=0}^{m} (-1)^{m-p} \binom{m}{p} a_{k+p} \right). \tag{5}$$

The product of two sequences (a_k) and (b_k) is defined as

$$(a_k) \cdot (b_k) = (a_k b_k).$$

The discrete Fourier transform operator $\mathscr{F}$, working on a sequence of length n, and its inverse are defined as follows:

$$\mathscr{F}(a_k) = \left(\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} a_j e^{-(2\pi i/n)jk} \right),$$

$$\mathscr{F}^{-1}(a_k) = \left(\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} a_j e^{(2\pi i/n)jk} \right).$$

It is easy to prove that for any sequence (a_k) one has $\mathscr{F}^{-1}\mathscr{F}(a_k) = (a_k)$. The only ingredient is the summation formula for geometric sequences. The operator $\mathscr{F}$ has some remarkable and extremely useful properties. If we define the (squared) norm of a sequence (a_k) to be the number

$$\|(a_k)\|^2 = \sum_{k=0}^{n-1} |a_k|^2,$$

then the Plancherel–Parseval theorem tells us that $\|\mathscr{F}(a_k)\|^2 = \|(a_k)\|^2$; the Fourier transform does not change the norm of a sequence. This is very useful: if we want to know how an operator changes the norm of a sequence, we can just as well look at how the sequence's norm changes in Fourier space, which sometimes is a great deal easier, as the following formula shows.

$$\mathscr{F}\Delta(a_k) = (e^{(2\pi i/n)k} - 1) \cdot \mathscr{F}(a_k) \tag{6}$$

or, equivalently,

$$\Delta(a_k) = \mathscr{F}^{-1}((e^{(2\pi i/n)k} - 1) \cdot \mathscr{F}(a_k)).$$

Proof. $\mathscr{F}\Delta(a_k) = (1/\sqrt{n}) \sum_{j=0}^{n-1} (a_{j+1} - a_j)\, e^{-(2\pi i/n) jk}) = ((1/\sqrt{n}) \sum_{j=0}^{n-1} (e^{(2\pi i/n)k} - 1)\, a_j e^{-(2\pi i/n) jk}) = (e^{(2\pi i/n)k} - 1) \cdot \mathscr{F}(a_k)$. So in Fourier space, the linear operator Δ is a diagonal operator: components of the Fourier transformed sequence do not mix under Δ. By applying Eq. (6) a number of times, we obtain

$$\mathscr{F}\Delta^d(a_k) = ((e^{(2\pi i/n)}k - 1)^d) \cdot \mathscr{F}(a_k).$$

The squared modulus of $e^{(2\pi i/n)j} - 1)(e^{-(2\pi i/n)j} - 1) = 2 - 2\cos(2\pi j/n)$. This, together with the above equality and the Plancherel–Parseval theorem gives

$$\|\Delta^d(a_k)\|^2 = \sum_{j=0}^{n-1} \left(2 - 2\cos\frac{2\pi j}{n}\right)^d |\mathscr{F}(a_k)_j|^2, \tag{7}$$

where $\mathscr{F}(a_k)_j$ means the jth component of the Fourier transform of (a_k).

The subject of finite Fourier transforms is treated in more detail in [7], starting at paragraph 4.4.

3. Proofs of the Inequalities

Now we have defined enough to write the lemma, on which the proofs of Eqs. (1)–(4) depend.

Lemma 1. *Let a sequence $(a_k) \in \mathbb{C}^n$ of length n be given, and let $d \in \mathbb{N}$ be greater than zero. Then the following free statements hold:*

(a) *If $\sum_{k=0}^{n-1} a_k = 0$, then*

$$\|\Delta^d(a_k)\|^2 \geqslant \left(2 - 2\cos\frac{2\pi}{n}\right)^d \|(a_k)\|^2,$$

with equality if and only if $(a_k) = \alpha(e^{(2\pi i/n)k}) + \beta(e^{-(2\pi i/n)k})$ for constants $\alpha, \beta \in \mathbb{C}$.

(b) *If n is even, then*

$$\|\Delta^d(a_k)\|^2 \leqslant 4^d \|(a_k)\|^2,$$

with equality if and only if $(a_k) = \alpha((-1)^k)$ for some constant $\alpha \in \mathbb{C}$.

(c) *If n is odd, then*

$$\|\Delta^d(a_k)\|^2 \leqslant \left(2 + 2\cos\frac{\pi}{n}\right)^d \|(a_k)\|^2,$$

with equality if and only if $(a_k)=\alpha(e^{(2\pi i/n)((n+1)/2)k})+\beta(e^{(2\pi i/n)((n-1)/2)k})$ *for constants* $\alpha, \beta \in \mathbb{C}$.

(d) *If n is even, and* $\sum_{k=0}^{n-1}(-1)^k a_k=0$, *then*

$$\|\Delta^d(a_k)\|^2 \leqslant \left(2+2\cos\frac{2\pi}{n}\right)^d \|(a_k)\|^2,$$

with equality if and only if $(a_k)=\alpha(e^{(2\pi i/n)(n/2+1)k})+\beta(e^{(2\pi i/n)(n/2-1)k})$ *for constants* $\alpha, \beta \in \mathbb{C}$.

(e) *If n is even, and* $\sum_{k=0}^{n-1} e^{(2\pi i/n)mk} a_k=0$ *for* $m=\frac{1}{2}n$, $\frac{1}{2}n-1$, *and* $\frac{1}{2}n+1$, *then*

$$\|\Delta^d(a_k)\|^2 \leqslant \left(2+2\cos\frac{4\pi}{n}\right)^d \|(a_k)\|^2,$$

with equality if and only if $(a_k)=\alpha(e^{(2\pi i/n)(n/2+2)k})+\beta(e^{(2\pi i/n)(n/2-2)k})$ *for constants* $\alpha, \beta \in \mathbb{C}$.

As is said above, (a–c) are proved in Schoenberg [5]. An elegant geometrical proof of (a) can also be found in Shisha [6].

Proof. (a) If $\|(a_k)\|^2$ is held constant (which is the same as keeping $\sum_{j=0}^{n-1}|\mathscr{F}(a_k)_j|^2$ constant), the sum on the right of Eq. (7) is a minimum when all the "mass" is put in the Fourier component or components with the least factor associated to it. This factor, $(2-2\cos(2\pi j/n))^d$, is zero when $j=0$, but by assumption the corresponding component $\mathscr{F}(a_k)_0=\sum_{k=0}^{n-1} a_k$ is also zero, so no mass can be put there. The minimum is therefore attained if and only if all mass is distributed over $\mathscr{F}(a_k)_1$ and $\mathscr{F}(a_k)_{n-1}$, which means that (a_k) should be a complex linear combination of $(e^{(2\pi i/n)k})$ and $(e^{-(2\pi i/n)k})$, and then $\|\Delta(a_k)\|^2=2(1-\cos(2\pi/n))\,\|(a_k)\|^2$. This proves the first part.

(b) From here, the inequalities are the reverse of the one in (a), so we seek for the maximum among the factors $2(1-\cos(2\pi j/n))$, but for the rest the proofs are completely analogous to the one above. Since n is even, the maximum among the factors $(2-2\cos(2\pi j/n))^d$ is 4^d, attained when $j=\frac{1}{2}n$, and the corresponding sequence is $((-1)^k)$. Therefore equality holds if and only if the sequence is of the form $(c(-1)^k)$ for some constant $c\in\mathbb{C}$.

(c) Now n is odd, so $(2-2\cos(2\pi j/n))^d$ attains its maximum when $j=\frac{1}{2}(n+1)$ or $j=\frac{1}{2}(n-1)$, where it reaches the value $(2-2\cos(2\pi\cdot\frac{1}{2}(n\pm 1)/n))^d=(2+2\cos(\pi/n))^d$. So the inequality is proved, and the sequences for which equality holds are the complex linear combinations of $e^{(2\pi i/n)((n+1)/2)k})$ and $(e^{(2\pi i/n)((n-1)/2)k})$.

(d) Here n is again even, but since it now is given that $\sum_{k=0}^{n-1}(-1)^k a_k=0$, that is, $\mathscr{F}(a_k)_{n/2}=0$, the sequence that resulted in part

(b) is no longer allowed. The maximum of the $(2-2\cos(2\pi j/n))^d$'s, given that $j=n/2$ is forbidden, is now attained when $j=n/2\pm 1$, whereby the inequality is proved. The extremising sequences are those that are linear combinations of $(e^{(2\pi i/n)(n/2+1)k})$ and $(e^{(2\pi i/n)(n/2-1)k})$.

(e) This is almost the same as case (d), with the difference that not only the $(n/2)$'th Fourier component should be zero, but also the $j=(n/2\pm 1)$'th. This leads to the required inequality and extremising sequences. This proves the lemma.

The reason that the five on first sight rather haphazardly chosen inequalities above have been proved is that some of them are needed for the proof of the inequalities (1)–(4), which I give in a moment. (The others were too beautiful to leave behind!)

The inequalities proven above cannot be applied directly to prove Eqs. (1)–(4). The problem is that the boundary conditions, for instance, that the first and the last of the numbers is zero or that the numbers are real, have to be incorporated in some way. The outcome depends in an essential way on these boundary conditions. For example, if in (1) it were not demanded that the first and last number be zero, the result would be that the minimum sequence is the constant sequence, because for each constant sequence one has $\Delta(a_k)=(0)$.

The proofs below all (but one) have the same structure. Given a real sequence $b_1, ..., b_n \in \mathbb{R}^n$ with boundary conditions, this sequence is inbedded in $\mathbb{C}^m$ as a complex (periodic) sequence (a_k), with $m>n$. This inbedding is chosen in such a way that $\|(a_k)\|^2$ is an integral number times $\sum_i |b_i|^2$, and also that $\|\Delta(a_k)\|^2$ is an integer times $\sum_i |b_i-b_{i-1}|^2$. Then some part of Lemma 1 is applied (with $d=1$, since we are dealing with first differences only for the moment). This gives a linear minimum-sequence-space (or maximum-, of course) of dimension one or two, i.e., linear combinations of one or two sequences, that minimize (or maximize) $\|\Delta(a_k)\|^2$ divided by $\|(a_k)\|^2$. We then check that this space contains a subspace that contains some inbedded real requence of the form we started with. The corresponding real sequences $b_1, ..., b_n$ therefore minimize $\sum_i |b_i-b_{i-1}|^2$ (under fixed $\sum_i |b_i|^2$), because these sums only differ by a factor to the norms of the corresponding complex sequences.

Proof of Eqs. (1)–(4). (1) Inbed the sequence of real numbers b_i in $\mathbb{C}^{2n+2}$ by associating with the numbers b_i the sequence $(a_k)=(b_0, b_1, ..., b_n, -b_0, -b_1, ..., -b_n)$. It is obvious that $\sum_{k=0}^{(2n+2)-1} a_k=0$, so Lemma 1(a) applies and yields

$$\|\Delta(a_k)\|^2 \geqslant 2\left(1-\cos\frac{2\pi}{2n+2}\right)\|(a_k)\|^2 = 2\left(1-\cos\frac{\pi}{n+1}\right)\|(a_k)\|^2.$$

Now $\|(a_k)\|^2 = 2\sum_{k=1}^{n} |b_k|^2$ and $\|\Delta(a_k)\|^2 = 2\sum_{k=1}^{n} |b_k - b_{k-1}|^2$, the last equality holding because $a_0 = a_{n+1}$. So the above inequality directly gives the desired result if we can show that among the sequences that minimize the left-hand side there are some that can be obtained by inbedding a real sequence b_i in $\mathbb{C}^{2n+2}$. First of all, the result of inbedding a real requence is obviously real. The minimizing sequences are of the form $\alpha(e^{(2\pi i/(2n+2))k}) + \beta(e^{-(2\pi i/(2n+2))k})$; the real sequences among these are exactly those of the form $\alpha \sin((\pi/(n+1))k + \gamma)$, with $\alpha, \gamma \in \mathbb{R}$. Inbedded sequences also have their zeroth coefficient zero, so γ must be zero. This sequence can be obtained by inbedding the real number sequence $b_i = \alpha \sin(k\pi/(n+1))$, which proves Eq. (1).

(2) Equation (2) follows from the first, and does not use Lemma 1 directly. Inbed the given sequence in a *real*-number sequence (c_i), whose components are given by $(c_i) = (b_0 = 0, b_1, ..., b_{n-1}, b_n, b_n, b_{n-1}, ..., b_1, b_0 = 0) \in \mathbb{R}^{2n}$. It is clear that $\sum_{k=1}^{2n} |c_k|^2 = 2\sum_{k=1}^{n} |b_k|^2$ and also that $\sum_{k=1}^{2n+1} |c_k - c_{k-1}|^2 = 2\sum_{k=1}^{n} |b_k - b_{k-1}|^2$ because $|c_n - c_{n-1}|^2 = 0$. The sequence satisfies the condition that $c_0 = c_{2n+1} = 0$, so Eq. (1) applies and yields

$$\sum_{k=1}^{2n+1} (c_k - c_{k-1})^2 \geq 2\left(1 - \cos\frac{\pi}{2n+1}\right) \sum_{k=1}^{n} c_k^2,$$

with equality if $c_k = \alpha \sin(k\pi/(2n+1))$. For these numbers it is true that $c_{n-k} = c_{n+1+k}$, so this is indeed an inbedded sequence. This proves Eq. (2).

(3) Real numbers $b_1, ..., b_n$ are given, and $b_0 = b_{n+1} = 0$. Inbed this sequence in $(a_k) = (b_0, b_1, ..., b_n, (-1)^n b_0, (-1)^n b_1, ..., (-1)^n b_n) \in \mathbb{C}^{2n+2}$. Now $\sum_{k=0}^{2n+1} (-1)^k a_k = 0$, so Lemma 1(d) applies and states that

$$\|\Delta(a_k)\|^2 \leq 2\left(1 + \cos\frac{2\pi}{2n+2}\right) \|(a_k)\|^2,$$

with equality if and only if $(a_k) = \alpha(e^{(2\pi i/(2n+2))((2n+2)/2+1)k}) + \beta(e^{(2\pi i/(2n+2))((2n+2)/2-1)k})$. For a sequence of this form to be the result of an inbedding, it should be at least real and a_0 should be zero; therefore $(a_k) = (\alpha(-1)^k \sin((\pi/(n-1))k))$, and it is easily checked that this is an inbedded sequence. This proves (3).

(4) Given are real numbers $b_1, ..., b_n$ and $b_0 = 0$. Inbed this sequence in the sequence (a_k) in the following way:

$$(a_k) = (b_0, b_1, ..., b_n, b_n, b_{n-1}, ..., b_1, -b_0, -b_1, ..., -b_n, -b_n, -b_{n-1}, ..., -b_1) \in \mathbb{C}^{4n+2}.$$

First, note that $\|(a_k)\|^2 = 4\sum_{k=1}^{n} |b_k|^2$, and that $\|\Delta(a_k)\|^2 = 4\sum_{k=1}^{n} |b_k - b_{k-1}|^2$, the last equality holding because $a_n = a_{n+1}$ and $a_{2n+1} = -b_0 = b_0$. It is clear that $\sum_{k=0}^{4n+1} (-1)^k a_k = 0$: the first quarter of the sequence vanishes against the second, and the third against the fourth. Furthermore, $\sum_{k=0}^{4n+1} (-1)^k e^{\pm(2\pi i/(4n+2))k} a_k = 0$ because, since $a_k = -a_{k+2n+1}$,

$$\begin{aligned} &a_k(-1)^k e^{\pm(2\pi i/(4n+2))k} + a_{k+2n+1}(-1)^{k+2n+1} e^{\pm(2\pi i/(4n+2))(k+2n+1)} \\ &\quad = a_k(-1)^k e^{\pm(2\pi i/(4n+2))k} + a_{k+2n+1}(-1)^k e^{\pm(2\pi i/(4n+2))k} = 0, \end{aligned}$$

so the sum of all those pairs is also zero. Thus part (e) of the lemma may be applied. It yields

$$\|\Delta(a_k)\|^2 \leqslant 2\left(1 + \cos\frac{4\pi}{4n+2}\right) \|(a_k)\|^2,$$

with equality if and only if $(a_k) = \alpha(e^{(2\pi i/(4n+2))((4n+2)/2+2)k}) + \beta(e^{(2\pi i/(4n+2))((4n+2)/2-2)k})$. All real sequences with zeroth coefficient zero among these are of the form

$$(a_k) = \left(\alpha(-1)^k \sin\frac{2k\pi}{2n+1}\right),$$

(with α real) which can easily be checked to be an inbedded sequence. This proves Eq. (4).

The case of second differences is similar to that of first differences. The following result answers the question raised by Alzer in [3].

THEOREM. *Let n real numbers $b_1, ..., b_n$ be given, and let $b_0 = b_{n+1} = 0$. Then the following inequalities hold*:

$$\begin{aligned} \left(2\sin\frac{\pi}{2(n+1)}\right)^4 \sum_{k=1}^{n} b_k^2 &\leqslant \sum_{k=0}^{n-1} (b_k - 2b_{k+1} + b_{k+2})^2 \\ &\leqslant \left(2 + 2\cos\frac{\pi}{n+1}\right)^2 \sum_{k=1}^{n} b_k^2. \end{aligned}$$

The left inequality is attained if and only if $b_k = \alpha\sin(k\pi/(n+1))$ for some $\alpha \in \mathbb{R}$, and the right one is attained if and only if $b_k = \alpha(-1)^k \sin(k\pi/(n+1))$ for some $\alpha \in \mathbb{R}$.

The left inequality was already proved by Fan *et al.* in [1].

Proof. Inbed the sequence of numbers $b_1, ..., b_n$ in the complex-valued periodic sequence $(a_k) \in \mathbb{C}^{2n+2}$ by defining $(a_k) = (0, b_1, ..., b_{n-1}, b_n,$

$0, -b_n, -b_{n-1}, ..., -b_1)$. If we write $\delta_k = b_k - 2b_{k+1} + b_{k+2}$ for $k = 0, ..., n-1$, then, since $b_0 = b_{n+1} = 0$, $\Delta^2(a_k) = (\delta_0, ..., \delta_{n-1}, 0, -\delta_{n-1}, ..., -\delta_0, 0)$, so that $\|\Delta^2(a_k)\|^2 = 2\sum_{k=0}^{n-1} \delta_k^2$, and it is obvious that $\|(a_k)\|^2 = 2\sum_{k=1}^{n} b_k^2$. Therefore, we should minimize and maximize the norm $\|\Delta^2(a_k)\|$ relative to $\|(a_k)\|$ among the sequences (a_k) of the above form.

The sequences (a_k) constructed above always obey the condition $\sum_{k=0}^{2n+1} a_k = 0$, so we can apply part (a) of Lemma 1 (with "n" $= 2n+2$),

$$\|\Delta^2(a_k)\|^2 \geqslant \left(2 - 2\cos\frac{2\pi}{2n+2}\right)^2 \|(a_k)\|^2,$$

with equality if and only if $(a_k) = \alpha(e^{(2\pi i/(2n+2))k}) + \beta(e^{-(2\pi i/(2n+2))k})$ for constants $\alpha, \beta \in \mathbb{C}$. Among these sequences, only those equal to $(\alpha \sin(k\pi/(n+1)))$ for some $\alpha \in \mathbb{R}$ are of the required form. And because $2 - 2\cos(2\pi/(2n+2)) = 4\sin^2(\pi/2(n+1))$, this proves the left inequality.

The sequence (a_k) also obeys $\sum_{k=0}^{2n+1} (-1)^k a_k = 0$, so part (d) of Lemma 1 is applicable, and this time yields

$$\|\Delta^2(a_k)\|^2 \leqslant \left(2 + 2\cos\frac{2\pi}{2n+2}\right)^2 \|(a_k)\|^2,$$

with equality if and only if $(a_k) = \alpha(e^{(2\pi i/(2n+2))((2n+2)/2+1)k}) + \beta(e^{(2\pi i/(2n+2))((2n+2)/2-1)k})$ for constants $\alpha, \beta \in \mathbb{C}$. The real-valued first-coefficient-zero sequences among these are precisely those with $(a_k) = (\alpha(-1)^k \sin(k\pi/(n+1)))$ for some $\alpha \in \mathbb{R}$, and these sequences are indeed of the required form. This proves the right inequality, and thereby the theorem.

If the requirement that b_{n+1} be zero is lifted and no other is imposed, no sensible generalisation of the theorem is obtained. The linear sequence $b_k = \alpha k$ for instance can have a norm as high as you wish and still have zero second difference.

A different way of generalising the above double inequality is to ask whether a similar inequality exists for third differences. That is, what are the best possible constants A and B such that

$$A\sum_{k=1}^{n} b_k^2 \leqslant \sum_{k=0}^{n-2} (b_k - 3b_{k+1} + 3b_{k+2} - b_{k+3})^2 \leqslant B\sum_{k=1}^{n} b_k^2$$

for all real numbers $0 = b_0, b_1, ..., b_n, b_{n+1} = 0$? The constant A is zero, because the third difference of $b_k = k(n+1-k)$ is identically zero. The value of B for different n is unknown. The method used in this paper for proving the double inequality for second differences cannot be used to find the constant in this case: it depends on an appropriate inbedding in a sequence space without boundary conditions. That such an inbedding does not exist anymore in this case is very probable if you look at the form of the minimizing sequence for the left inequality: instead of a sine, it is a polynomial.

Finally, it is an interesting, open question whether an elementary proof exists of the double inequality in the theorem above, similar to the proofs of (1)–(4) in [3] and [4].

4. EIGENVALUES AND EIGENVECTORS OF ROTATION-INVARIANT MATRICES

Let us now turn to the complex-valued periodic sequences. Suppose m complex constants $\beta_0, ..., \beta_{m-1}$ are given, ($m \leqslant n$), and one is to tell how to choose the sequence $(a_k) \in \mathbb{C}^n$ so that the norm of the sequence $A(a_k)$ is minimal, or maximal, with respect to $\|(a_k)\|$, where the linear rotation-invariant operator A is defined by

$$A(a_k) = \left(\sum_{j=0}^{m-1} \beta_j a_{k+j} \right).$$

(The operator A is called rotation-invariant because it commutes with rotations, or shifts.) This problem is easily solved by using discrete Fourier transformations.

First, find constants $\gamma_0, ..., \gamma_{m-1}$ so that

$$\left(\sum_{p=0}^{m-1} \beta_p a_{k+p} \right) = \sum_{q=0}^{m-1} \gamma_q \Delta^q(a_k)$$

Using Eq. (5), and equating all the terms with equal displacement p, this is equivalent to the condition

$$\beta_p = \sum_{q=p}^{m-1} (-1)^{q-p} \binom{q}{p} \gamma_q.$$

This is a system of m linear equations in m unknowns. To find the solution, what is needed is the inverse of the matrix

$$\left((-1)^{q-p} \binom{q}{p} \right)_{pq},$$

where the indices p, q run through $0, ..., n-1$ (instead of the more usual range $1, ..., n$). For example, when $n = 4$, what is needed is the inverse of the matrix

$$\begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

which in this case is

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The inverse of the general n by n matrix is, as might be expected from the above example, the same matrix except that all elements are positive.

LEMMA 2.

$$\left((-1)^{q-p}\binom{q}{p}\right)_{pq}^{-1} = \left(\binom{q}{p}\right)_{pq},$$

where the indices p, q *run through* $0, ..., n-1$.

For the proof of this lemma I refer to the final paragraph. Because the β's are given, the γ's can now be calculated by letting the above matrix operate on the vector of the β's. We thus obtain the representation of the operator A in terms of powers of Δ. By Eq. (6) and the linearity of $\mathscr{F}$ and Δ, we have

$$\begin{aligned} \mathscr{F}A(a_k) &= \mathscr{F}\left(\sum_{d=0}^{m-1} \gamma_d \Delta^d\right)(a_k) \\ &= \left(\sum_{d=0}^{m-1} \gamma_d (e^{(2\pi i/n)k} - 1)^d\right) \cdot \mathscr{F}(a_k) = (P(e^{(2\pi i/n)k} - 1)) \cdot \mathscr{F}(a_k), \end{aligned}$$

where P is the polynomial $P(x) = \gamma_0 + \gamma_1 x + \cdots + \gamma_{m-1} x^{m-1}$. This means that

$$\|A(a_k)\|^2 = \|(P(e^{(2\pi i/n)k} - 1)) \cdot \mathscr{F}(a_k)\|^2.$$

To minimize or maximize the left-hand-size, j should be chosen so that $|P(e^{(2\pi i/n)j} - 1)|^2$ is minimal, respectively maximal, and (a_k) must then be set equal to $(\alpha e^{(2\pi ij/n)k})$. It can happen that there are several values of j that make $|P(e^{(2\pi i/n)j} - 1)|^2$ minimal or maximal. In that case the minimizing or maximizing sequences form a linear space, with dimension equal to the number of values of j found.

The sequences $(e^{(2\pi ij/n)k})$, $j=0, ..., n-1$, are "eigensequences" of the operator Δ: if Δ operates on the sequence it does not change except that it is multiplied with some factor (see Eq. (6) and note that the Fourier transform of $(e^{(2\pi ij/n)k})$ is the sequence that is zero everywhere except at the jth place). If the complex-valued periodic sequences are identified with vectors in $\mathbb{C}^n$, and the linear rotation-invariant operator A with a matrix $A=(a_{ij})\in\mathbb{C}^{n\times n}$, then the above discussion is a proof of the following theorem.

THEOREM. *Let $A=(a_{ij})\in\mathbb{C}^{n\times n}$ be a rotation-invariant matrix, that is, $a_{ij}=a_{(i+1)(j+1)}$ for every i and j, where the indices are counted modulo n. Then the eigenvectors of the matrix are*

$$e_j=(1, e^{2\pi ij/n}, ..., e^{(2\pi ij/n)(n-1)})^t, \qquad j=0, ..., n-1,$$

(where the "t" denotes transposition) and the eigenvalues corresponding to these eigenvectors are the elements of the vector

$$((e^{2\pi ip/n}-1)^q)_{pq}\left(\binom{r}{q}\right)_{qr}(a_{00}, a_{01}, ..., a_{0(n-1)})^t.$$

The theorem can also be proved directly by writing out the matrix product:

$$\sum_{q=0}^{n}(e^{2\pi ip/n}-1)^q\binom{r}{q}=((e^{2\pi ip/n}-1)+1)^r=e^{2\pi ipr/n},$$

so the product of the two matrices is exactly the matrix of eigenvectors of A. Applying it to the transposed first row of A therefore yields the vector of eigenvalues multiplied with the first component of each eigenvector, which are all 1.

5. Proof of Lemma 2

The following is to be proved.

$$\sum_{j=0}^{n-1}(-1)^{j-p}\binom{j}{p}\binom{q}{j}=\delta_{pq},$$

where the Kronecker symbol δ is defined by $\delta_{pq}=1$ if $p=q$, and $\delta_{pq}=0$ otherwise. The proof runs by induction on q. For $q=0$, the equality is simple: if $p=0$ the only nonzero term is the $j=0$ term which equals one, and if $p\geqslant 1$ all terms are zero. So now suppose the equality is true for some $q<n-1$ and all p. Then

$$\begin{aligned}\sum_{j=0}^{n-1} &(-1)^{j-p}\binom{j}{p}\binom{q+1}{j}\\ &=\sum_{j=0}^{n-1}(-1)^{j-p}\binom{j}{p}\left(\binom{q}{j-1}+\binom{q}{j}\right)\\ &=\sum_{j=0}^{n-1}(-1)^{j-p}\binom{j}{p}\binom{q}{j}\\ &\quad+\sum_{j=0}^{n-1}(-1)^{j-p}\left(\binom{j-1}{p-1}+\binom{j-1}{p}\right)\binom{q}{j-1}.\end{aligned}$$

By the induction hypothesis, the first term is just δ_{pq}. Since $q<n-1$, $\binom{q}{-1}=\binom{q}{n-1}=0$ so that in the second term the range of the summation can be changed from $0, ..., n-1$ into $1, ..., n$. Substituting j by $j+1$, we get

$$\begin{aligned}\delta_{pq}&+\sum_{j=0}^{n-1}(-1)^{j-(p-1)}\binom{j}{p-1}\binom{q}{j}+\sum_{j=0}^{n-1}(-1)^{j-p+1}\binom{j}{p}\binom{q}{j}\\ &=\delta_{pq}+\delta_{p-1,q}-\delta_{pq}=\delta_{p-1,q}=\delta_{p,q+1},\end{aligned}$$

where the induction hypothesis was used twice. This proves the lemma.

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